

Charge Density Bounds in Superconducting States of Strongly Correlated Systems

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Charge density bounds of knotted and linked vortex states in two-component Ginzburg-Landau model are considered. When the mutual linking number of vector order parameter vortex lines is less than the Hopf invariant, these states have the lower-lying energies. It is shown that a set of local minima of free energy contains new classes of universality which exist in a hole density range limited at both finite ends.

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Introduction. – A tangle of vortex filaments is a system which attracts attention due to several reasons. Along with the coherent state, which is the background of this vortex field distributions, the filament system also contains a disorder combination due to free motion of its fragments and a topological order because of the effects of knotting and linking of its separate parts. The degree of this order can be characterized by the linking number of vortices. Persistent linked field configurations along with a large value of interaction energy present the basic content of what we mean when talking about the origin of strong correlations.

The study of soft condensed matter, whose universal behavior is determined by topological characteristics, is an active area of research in physics [1, 2, 3, 4] and beyond [5]. Indeed, the dynamics of entangled vortex states is recognized as one of the most challenging problems of modern condensed matter physics. The emergence of novel analytical methods [2, 3, 4] to treat nonlinear field equations create now an unique opportunity to understand the dynamics of entangled vortex states.

The vortex dynamics can be understood in detail in the framework of basic models, such as the Ginzburg-Landau functional or more complicated multi-component systems. Main theoretical approaches for analyzing universal behavior in these models are based on the methods of the topological field theory [6] and nonlinear dynamics [7]. The universality classes are defined both by symmetry and topological characteristics of the background nonlinear fields, while a system with infinite number of degrees of freedom contains in the ground state only the finite number of symmetrically invariant physical states.

The aim of this paper is to elucidate the physical mechanism leading to formation of the charge density range bounds for superconducting states with a set of numbers determining the knotting and linking degree of the fields that take part in the description of the coherent state. The use of the Ginzburg-Landau functional and the methods of the topological gauge field theory gives us the hope that the obtained answers are universal. A combination of novel methods will make it possible to advance in description of the processes which are determined by the contribution of topological excitations.

Model. – We will use the Ginzburg-Landau free energy

$$F = \int d^3x \left[\sum_{\alpha} \frac{1}{2m} \left| \left(\hbar \partial_k + i \frac{2e}{c} A_k \right) \Psi_{\alpha} \right|^2 + \sum_{\alpha} \left(-b_{\alpha} |\Psi_{\alpha}|^2 + \frac{c_{\alpha}}{2} |\Psi_{\alpha}|^4 \right) + \frac{\mathbf{B}^2}{8\pi} \right] \quad (1)$$

with a two-component order parameter

$$\Psi_{\alpha} = \sqrt{2m} \rho \chi_{\alpha}, \quad \chi_{\alpha} = |\chi_{\alpha}| e^{i\varphi_{\alpha}}, \quad (2)$$

satisfying the CP^1 condition, $|\chi_1|^2 + |\chi_2|^2 = 1$. This model is used in the context of the two-gap superconductivity [8] and in the non-Abelian field theory [9, 10].

It has been shown in paper [8] that there exists an exact mapping of the model (1), (2) into the following version of \mathbf{n} -field model:

$$F = \int d^3x \left[\frac{1}{4} \rho^2 (\partial_k \mathbf{n})^2 + (\partial_k \rho)^2 + \frac{1}{16} \rho^2 \mathbf{c}^2 + (F_{ik} - H_{ik})^2 + V(\rho, n_3) \right]. \quad (3)$$

To write down Eq. (3), dimensionless units and gauge invariant order parameter fields of the unit vector $\mathbf{n} = \bar{\chi} \boldsymbol{\sigma} \chi$, and the velocity $\mathbf{c} = \mathbf{J}/\rho^2$ have been used. Here $\bar{\chi} = (\chi_1^*, \chi_2^*)$ and $\boldsymbol{\sigma}$ - Pauli matrices. The effective coupling constant $512e^2/\hbar c$ in this case [8] has the order of unity. The full current $\mathbf{J} = 2\rho^2(\mathbf{j} - 4\mathbf{A})$ has a paramagnetic ($\mathbf{j} = i[\chi_1 \nabla \chi_1^* - c.c. + (1 \rightarrow 2)]$) and a diamagnetic ($-4\mathbf{A}$) parts. Besides, in Eq. (3) $F_{ik} = \partial_i c_k - \partial_k c_i$, $H_{ik} = \mathbf{n} \cdot [\partial_i \mathbf{n} \times \partial_k \mathbf{n}] := \partial_i a_k - \partial_k a_i$.

Setting in Eq. (3) $\mathbf{c} = 0$ we come back to the model of [11]. The numerical study of the knotted configurations of \mathbf{n} -field in this model has been done in [12, 13, 14]. The lower energy bound in this case

$$F \geq 32\pi^2 |Q|^{3/4} \quad (4)$$

is determined [15, 16, 17] by the Hopf invariant,

$$Q = \frac{1}{16\pi^2} \int d^3x \varepsilon_{ikl} a_i \partial_k a_l. \quad (5)$$

At compactification $R^3 \rightarrow S^3$ and $\mathbf{n} \in S^2$, the integer $Q \in \pi_3(S^2) = \mathbb{Z}$ shows the degree of linking or knotting of filamental manifolds $\mathcal{M} \in S^3$, where the vector field $\mathbf{n}(x, y, z)$ is defined. In particular, for two linked rings (Hopf linking) $Q=1$, for the trefoil knot $Q=6$ and etc. Significant point [18] is as follows: $\pi_3(CP^M) = 0$ at $M > 1$ and $\pi_3(CP^1) = \pi_3(S^2) = \mathbb{Z}$. In the latter case the order parameter (2) is two-component one [8] and linked or knotted soliton configurations are labeled by the Hopf invariant (5).

Energy bounds. – Let us assume that $\rho = \rho_0$ can be find from the minimal value $V(\rho_0)$ of the potential $V(\rho)$, but the velocity \mathbf{c} does not equal zero. Equation (3) in this case has the following form:

$$F = F_n + F_c - F_{int} = \int d^3x \left[\left((\partial_k \mathbf{n})^2 + H_{ik}^2 \right) + \left(\frac{1}{4} \mathbf{c}^2 + F_{ik}^2 \right) - 2F_{ik}H_{ik} \right]. \quad (6)$$

It is seen from Eq. (6) that a superconducting state with $|\mathbf{c}| \ll 1$ has the energy which is less than the minimum in Eq. (4). To find the lower free energy bound in the superconducting state with $\mathbf{c} \neq 0$, it has been proved [19] that the following inequality for free energy parts takes place

$$F_n^{5/6} F_c^{1/2} \geq (32\pi^2)^{4/3} |L|, \quad (7)$$

where

$$L = \frac{1}{16\pi^2} \int d^3x \varepsilon_{ikl} c_i \partial_k a_l \quad (8)$$

is the degree of mutual linking [21] of the velocity \mathbf{c} lines and of the magnetic field $\mathbf{H} = [\nabla \times \mathbf{a}]$ lines. Quantities like (5), (8) arise as first integrals in the theory of an ideal or barotropic fluid [21]. In general, the integrals in Eqs. (5), (8) could be defined by the asymptotic linking numbers [22].

It follows from the Schwartz-Cauchy-Bunyakovsky inequality that $2F_n^{1/2} \cdot F_c^{1/2} \geq F_{int}$. Setting the boundary value F_{int} into Eq.(6), we get $F_{min} = (F_n^{1/2} - F_c^{1/2})^2$. The minimum value $32\pi^2 Q^{3/4}$ of the function F_n and Eq. (7) lead [19] to

$$F \geq 32\pi^2 |Q|^{3/4} (1 - |L|/|Q|)^2. \quad (9)$$

The trivial case $Q = 0$ should be considered after the limit $L = 0$. Let us pay also an attention to the self-dual relation $F_n = F_c$ which follows from F_{min} .

In general, the Hopf-Chern-Simons matrix,

$$K_{\alpha\beta} = \frac{1}{16\pi^2} \int d^3x \varepsilon_{ikl} a_i^\alpha \partial_k a_l^\beta = \begin{pmatrix} Q & L' \\ L & Q' \end{pmatrix}. \quad (10)$$

can be defined in our case by linking numbers $Q \in \mathbb{Z}$ of spin and spin-charge ($\{L, Q'\} \notin \mathbb{Z}$ at all) degrees of

freedom. In Eq. (10), $K_{\alpha\beta}$ is a symmetrical matrix ($L' = L$) [22] and $a_i^1 \equiv a_i$, $a_i^2 \equiv c_i$. Following the approach of [19] which is based on employ of the Hölder inequality chain as well as on the Ladyzhenskaya [20] inequality, one can find that

$$F_n^{1/2} F_c^{5/6} \geq (16\pi^2)^{4/3} |L'|. \quad (11)$$

The distinction of the coefficient in Eq. (11) from Eq. (7) results from the charge $2e$ of pairs (due to the coefficient $1/4$ in the first term of the free energy part F_c).

It follows from Eq. (9) that for all numbers $L < Q$ the energy of the ground state is less than that in the model described in [11], for which the inequality (4) is valid. The origin of the energy decrease can be easily understood. Even under the conditions of the existence of the paramagnetic part \mathbf{j} of the current \mathbf{J} , the diamagnetic interaction in the superconducting state consumes its own energy of the current and a part of the energy relating to the \mathbf{n} -field dynamics for all state classes with $L \leq Q$.

Charge density. – In the context of the high-temperature superconductivity problems [23, 24, 25, 26], Eq. (2) has the sense of the condition of the correlated or linked factorization of charge and spin degrees of freedom (from the gauge invariant order parameter point of view (3)). In fact, the constant value of the charge density plays the role of a tuning parameter of the system. In this case, the order parameter \mathbf{n} describes the distribution of spin degrees of freedom, while the order parameter \mathbf{c} contains the contribution both of spin degrees of freedom χ_α in the paramagnetic part of the current and of the $U(1)$ charge degrees of freedom in the diamagnetic part of the current \mathbf{J} . Due to such mixing, the gauge invariant order parameters \mathbf{n} and \mathbf{c} differ principally and topologically. If the mutual linking index is defined by the current \mathbf{J} , the measure in Eq. (8) depends on the charge density. The vector $\mathbf{c} = \mathbf{J}/\rho^2$ is normalized to the charge density and, as distinct from \mathbf{n} , belongs to the noncompact manifold. This leads to the fact that the Hopf numbers for it are not integer valued ($(L, Q') \notin \mathbb{Z}$), and the linking numbers are included into the factor of Eq. (9) in the form of the ratio L/Q .

Let the parameter ρ_0 change in some range. Since all terms in Eq. (3) are of the same order, the velocity \mathbf{c} and therefore the number L decrease as ρ_0 increases. In this case, if L is rather small, the smallest superconducting gap goes down with the growth of Q at the background of large value $32\pi^2 Q^{3/4}$ of the spin gap.

As ρ_0 decreases, the following effect takes place. The radius R of the compactification $R^3 \rightarrow S^3$ being proportional to ρ_0^{-1} , increases till it exceeds a certain critical value, R_{cr} . At $R > R_{cr}$ the Hopf map [27] is unstable relative to infinitesimal perturbation of vortex linked field distributions. As a result, the $U(2)$ symmetry associated with identical Hopf map is spontaneously broken.

This means that the topological solitons, instead of being spread out over the whole of S^3 , are then localized around a particular point (the base point of the stereographic projection) and collapse into localized structures [17]. This picture corresponds to the existence of an optimal value of the relation L/Q . Under such a key condition of the restoration of compactness of the base manifold, inhomogeneous superconducting state appears. As the parameter ρ_0 decreases further, the free energy F due to the inhomogeneity term $(\partial_k \rho)^2$ increases again and suppresses the superconducting gap.

Up to now the vector \mathbf{A} characterized the internal charge $U(1)$ gauge symmetry. If we apply an external electromagnetic field the vector potential \mathbf{A} equals the sum of the internal and the external gauge potentials. As a result, due to the diamagnetism of the superconducting state, the velocity \mathbf{c} decreases. Like in the case of increasing ρ_0 , this leads to suppression of the superconducting gap. The answer to the question on the existence of full or partial Meissner screening in these states depends on the result of the competition of contributions from neutral \mathbf{j} and charged $-4\mathbf{A}$ parts to the full current \mathbf{J} .

The soft case $\rho \neq \text{const}$ both for $\mathbf{c} = 0$ [29] and $\mathbf{c} \neq 0$ arouses certain interest. It is more complicated due to some reasons and will be considered in a separate paper. The point $\sqrt{F_c/F_n} = 1/4$ where the fields of the inequalities (7), (11) intersect, deserve an additional careful study as well. The equality in Eq. (9) under this remarks should be understood as an ideal limit depending on topological characteristics of knots and links only.

Discussion. – In the $(3+0)D$ case of the free energy (3), Hopf invariant (5) is analogous to the Chern-Simons action $(k/4\pi) \int dt d^2x \varepsilon_{\mu\nu\lambda} a_\mu \partial_\nu a_\lambda$ determining strong correlations of $(2+1)D$ modes [30, 31] at $k \simeq 2$. In planar systems, this coefficient has the sense of braiding of the excitation world lines. In particular, for the semion $k = 2$. Keeping in mind the relation of spatial dimensionality of the systems in their quantum and statistical descriptions, we note that the $(2+1)D$ dynamical case $k = 2$ of the open world line ends of excitations is equivalent to the compact $(3+0)D$ statistical example of the Hopf linking $Q = 1$ in Eq. (3) (see Fig. 1).

Similarly to the description of the states in the fractional quantum Hall effect with the filling factor $\nu = p/q$ and $p, q \in \mathbb{Z}$, the Eq. (9) depends on L/Q . The Hopf invariant $Q \in \mathbb{Z}$ enumerating gauge vakuua [28], is equivalent to the degree q of the ground state degeneracy and L plays the role of the filling degree p of the incompressible charged fluid state. We have stressed already that the number L is, generally, not integer valued. From this point of view the multiplier $(1 - L/Q)$ in Eq. (9) is equivalent to the Hall filling factor $1 - \nu$ for holes. (Let us remark that the mentioned parameter ρ_0^2 has the sense of the particle density). The distinction of our system from the fractional quantum Hall effect state is that due

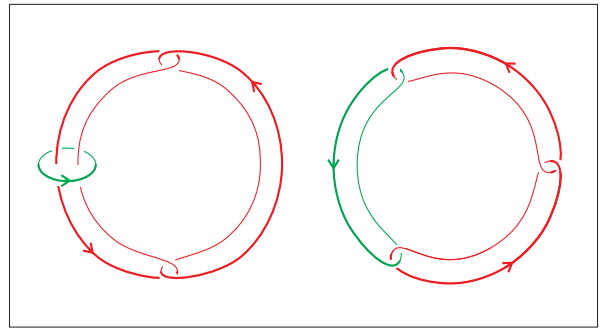


FIG. 1: Examples of \mathbf{n} - and \mathbf{c} -field line configurations in the form of Borromean rings with vanishing pairwise linking numbers $K_{\alpha\beta}$ (on the l.h.s.) and in the form of the closed chain of correlated charge-spin semion loops with $Q = 1$ (on the r.h.s.).

to compressibility of the superconducting state, where the charge gauge $U(1)$ symmetry is broken, the effective number L of the charge degrees of freedom is continuous. It plays the role of an interpolation parameter [32] which connects sectors with $L \notin \mathbb{Z}$ and $L \in \mathbb{Z}$ in one and the same universality class of Pfaffian states. Apparently it is true for all Pfaffian-like states with $Q > 1$ and numbers $L < Q$.

In conclusion, we have studied the charge density bounds of the superconducting states using the CP^1 version of Ginzburg-Landau model under the conditions of the existence of linking and knotting phenomena of the \mathbf{n} - and \mathbf{c} -fields being the gauge invariant order parameters of the considered system. We have shown that the superconducting states exist in the finite range $\rho_1^2 < \rho_0^2 < \rho_2^2$ of the charge density values. In this range, the field configurations with field lines characterized by semion values of the linking numbers are preferable.

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- [1] H.K. Moffatt, *Nature* **347**, 367 (1990).
 - [2] V.M.H. Ruutu, U. Parts, J.H. Koivuniemi, M. Krusius, E.V. Thuneberg, and G.E. Volovik, *JETP Lett.* **60**, 671 (1994).
 - [3] Yu.G. Makhlin, T.Sh. Misirpashaev, *JETP Lett.* **61**, 49 (1995).
 - [4] M.I. Monastyrsky, P.V. Sasorov, *Sov. Phys. JETP* **66**, 683 (1987).
 - [5] V. Katritch, J. Bednar, D. Michoud, R.G. Scharein, J. Dubochet, and A. Stasiak, *Nature* **384** 142 (1996).
 - [6] E. Witten, *Commun. Math. Phys.* **121**, 351 (1989).
 - [7] I. Aranson, L. Kramer, *Rev. Mod. Phys.* **74**, 99 (2002).

- [8] E. Babaev, L.D. Faddeev, A.J. Niemi, Phys. Rev. B **65**, 100512 (2002); cond-mat/0009438.
- [9] L.D. Faddeev, A.J. Niemi, Phys. Lett. **B525**, 195 (2002).
- [10] Y.M. Cho, Phys. Rev. Lett. **87**, 252001 (2001); hep-th/0110076.
- [11] L.D. Faddeev, A.J. Niemi, Nature **387**, 58 (1997).
- [12] J. Gładkowski, M. Hellmund, Phys. Rev. **D 56**, 5194 (1997).
- [13] R.A. Battye, P.M. Sutcliffe, Phys. Rev. Lett. **81**, 4798 (1998).
- [14] J. Hietarinta, P. Salo, Phys. Lett. **B451**, 60 (1999).
- [15] A.F. Vakulenko, L.V. Kapitansky, Sov. Phys. Dokl. **24**, 433 (1979).
- [16] A. Kundu, Yu.P. Rubakov, J. Phys. A **15**, 269 (1982).
- [17] R.S. Ward, Nonlinearity **12**, 1 (1999), hep-th/9811176.
- [18] A.G. Abanov, P.W. Wiegmann. Geometrical phases and quantum numbers of solitons in nonlinear sigma-models, hep-th/0105213.
- [19] A.P. Protogenov, V.A. Verbus, JETP Lett. (to be published).
- [20] O.A. Ladyzhenskaya. The mathematical theory of viscous incompressible flow, Gordon and Breach, 1969.
- [21] V.E. Zakharov, E.A. Kuznetsov, Phys. Usp. **40**, 1087 (1997).
- [22] V.I. Arnold, B.A. Khesin. Topological Methods in Hydrodynamics. Appl. Math. Sci. **125**, Chapt.3.
- [23] M. Sgrist, D.B. Bailey, and R.B. Laughlin, Phys. Rev. Lett. **74** 3249 (1995).
- [24] P.A. Lee, N. Nagaosa, T.K. Ng, and X.G. Wen, Phys. Rev. **B 57**, 6003 (1998).
- [25] P.A. Lee and X.G. Wen. Vortex structure in underdoped cuprates, cond-mat/0008419.
- [26] S. Chakravarty, R.B. Laughlin, D. Morr, C. Nayak. Hidden order in the cuprates, cond-mat/0005443.
- [27] E.A. Kuznetsov, A.V. Mikhailov, Phys. Lett. A **77** 37 (1980).
- [28] P. van Baal, A. Wipf. Classical gauge vacua as knots, hep-th/0105141.
- [29] M. Lübcke, S.M. Nasir, A. Niemi, and K. Torokoff, Phys. Lett. **B 534** 195 (2002); hep-th/0106102.
- [30] L.A. Abramyan, A.P. Protogenov, V.A. Verbus, JETP Lett. **69**, 887 (1999).
- [31] A.P. Protogenov, JETP Lett. **73**, 255 (2001).
- [32] N. Read, and Dmitry Green, Phys. Rev. B **61**, 10267 (2000).